ON THE QUASICONVEXITY OF THE FIXED SUBGROUP OF ENDOMORPHISMS OF RELATIVELY HYPERBOLIC GROUPS.

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Abstract. We prove that for any finitely generated relatively hyperbolic group G and any symmetric endomorphism ϕ of G with relatively quasiconvex image, Fixϕ is relatively quasiconvex subgroup of G.

1. Introduction

For a group G and an endomorphism ϕ, we denote by Fixϕ the subgroup of G that is fixed pointwise by ϕ. The subgroup Fixϕ has been studied by various authors and for various groups G (see [15] and references therein).

Relatively hyperbolic groups are groups that admit a geometrically finite action on a proper hyperbolic space. It is a large class of groups introduced by Gromov in its seminal paper [5] and it has attracted exceptional interest with very important results. The novice reader should consult the work of Bowditch [1, 2], the work of Tukia [14, 13] and of Osin [10] and Gerasimov [3] for different approaches. The rôle of relatively quasiconvex subgroups in the theory of relatively hyperbolic goups is as much central as that of quasiconvex subgroups in hyperbolic groups. Many authors have given different formulations for this idea, all of which turned out to be equivalent in the standard setup of countable relatively hyperbolic groups, as shown by Hruska [6].

Neumann [9] proved that the fixed subgroup of any automorphism of a hyperbolic group is quasiconvex. Recently, Minasyan and Osin [7] generalized Neumann's result by proving that if G is a finitely generated relatively hyperbolic group, then Fixϕ is relatively quasiconvex in the case of an automorphism ϕ of G that permutes the maximal parabolic subgroups.

If G is a relatively hyperbolic groups, then it is equivalent to say that G admits a geometrically finite convergence group action on a compact, metrizable space X. Following [12], we call an endomorphism ϕ of G *symmetric* if there exists a continuous map $\overline{\phi} : X \to X$ such that $\overline{\phi}(gx) = \phi(g)\overline{\phi}(x)$. The results in [12] can be used to obtain classes of relatively hyperbolic groups in which the fixed subgroup of any symmetric endomorphism is relatively quasiconvex. Therefore, it is very natural to look for generalizations of Minasyan and Osin's result to endomorphisms of relatively hyperbolic groups.

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In the present work we show that for any symmetric endomorphism ϕ of a geometrically finite group G the fixed subgroup Fixϕ is geometrically finite. Notice that if G is a finitely generated geometrically finite group, and ϕ is any automorphism which permutes maximal parabolic subgroups, then it follows from [2, Theorem 9.4] that ϕ induces a homeomorphism $\overline{\phi}$ of ΛG such that $\overline{\phi}(gx) = \phi(g)\overline{\phi}(x)$. Hence Fix$\phi$ is relatively quasiconvex and as an immediate consequence, we retrieve the result by Minasyan and Osin [7]. Notice also that using free products of locally quasiconvex relatively hyperbolic groups with finite groups, we can easily construct symmetric endomorphisms of relatively hyperbolic groups which are not even monomorphisms.

Our approach differs than this in [7]; it builds heavily on the work of Tukia in [14] and is based on the action of G on its Gromov boundary rather than its action on a hyperbolic model space.

2. Convergence groups and equivariant maps

We begin by recalling some facts about convergence groups. The reader could find more in the articles of Bowditch [2] (and in [1]), Tukia [13] and Gerasimov [3]. Suppose that X is a compact metric space and G a group that acts by homeomorphisms on X. We say that G is a *convergence group* if the induced action on the space T of distinct triples of X is properly discontinuous. We write $\Lambda G \subseteq X$ for the *limit set* of G, that is the unique minimal closed non-empty G-invariant subset of X. We may also assume that the action is *minimal*, in the sense that $\Lambda G = X$. So from now on, we identify X with ΛG and assume that X contains at least three points. Such an action is called *non-elementary*. Otherwise the action is *elementary* and in that case there is very little information we can get for G. For example, if ΛG contains only one point then G can be any countable group.

A subgroup of G is *parabolic* if it is infinite, fixes some point of X and contains no loxodromics. The fixed point is unique and called *parabolic*. The stabilizer of a parabolic point p is a parabolic subgroup H of G. If H acts cocompactly on $X \setminus \{p\}$, then we say that p is a *bounded parabolic point*. A point $p \in X$ is a *conical limit* point if there exist a sequence $g_i \in G$ and points $a \neq b$ in X such that $g_i p \to a$ but $g_i x \to b$ for any other $x \in X$. Conical limit points cannot be parabolic ([14, Theorem 3A]).

The (convergence) action of G is *geometrically finite (or cusp-uniform)* if every point of X is either conical or bounded parabolic. In that case G is a *relatively hyperbolic group* and there are finitely many orbits of bounded parabolic points [14]. The set of maximal parabolic subgroups is the *peripheral structure* of G. A subgroup H of G is called *relatively quasiconvex* if the induced convergence action of H on its limit set is geometrically finite. We refer the reader to [6] for equivalent characterizations of relative hyperbolicity.

Following Tukia we set $\overline{T} = T \cup X$ and define the topology of $\overline{T}$ so that T is an open subset of $\overline{T}$ and the distance between $z = (z_1, z_2, z_3) \in T$ and $x \in X$ is

$$d(z, x) = \text{the second largest of } \ d(z_i, x), \ i = 1, 2, 3$$

with the assumption that if two of $d(z_i, x)$ coincide then $d(z, x)$ is this distance. The above distance defines a metric that extends the metric of X. Notice

however, that d is not defined if both $x, z \in T$ (see [14, page 77]). Thus we can define an ε-neighbourhood of $x \in X$ to be the elements $\{z \in \overline{T} \mid d(z,x) < \varepsilon\}$. Then $\overline{T}$ is compact and the action of G on X and T induces a convergence action on $\overline{T}$, proper discontinuous on T.

Now let ϕ be a symmetric endomorphism of G, i.e. there exists a G-equivariant continuous map $\overline{\phi} : X \to X$, in the sense that $\overline{\phi}(gx) = \phi(g)\overline{\phi}(x)$. The map $\overline{\phi}$ can be extended naturally to a continuous map $\widehat{\phi} : T \to T$ such that $\widehat{\phi}(g(x_1, x_2, x_3)) = \phi(g)(\overline{\phi}(x_1), \overline{\phi}(x_2), \overline{\phi}(x_3))$. The fact that $\widehat{\phi}$ is continuous is an immediate consequence of the fact that T inherits the product topology of X. The G-equivariance of $\widehat{\phi}$ is a consequence of the definition of the action of G on T. This extension allow us to define a G-equivariant extension $\widetilde{\phi} : \overline{T} \to \overline{T}$ with

$$\widetilde{\phi}(x) = \begin{cases} \overline{\phi}(x) & \text{if } x \in X \\ \widehat{\phi}(x) & \text{if } x \in T \end{cases}$$

Lemma 1. *Under the above hypotheses, $\widetilde{\phi}$ is continuous and* $\Lambda \mathrm{Fix}\phi \subseteq \mathrm{Fix}\overline{\phi} = \{x \in X \mid \overline{\phi}(x) = x)\}$.

Proof. To show that $\widetilde{\phi}$ is continuous, it suffices to prove that $\widetilde{\phi}$ is continuous in each $x \in X$, since T is an open subset of $\overline{T}$. So let $x \in X$ and $\widetilde{\phi}(x) = \overline{\phi}(x)$ its image under $\widetilde{\phi}$. Let also $\varepsilon > 0$ and $V(\widetilde{\phi}(x), \varepsilon) = \{z \in \overline{T} \mid d(z,x) < \varepsilon\}$ an ε-neighbourhood of $\widetilde{\phi}(x)$. Since $\overline{\phi}$ is continuous, we have that for every $\varepsilon > 0$ there is a $\delta > 0$ such that the $\overline{\phi}$-image of the δ-neighbourhood of x in X is contained in the ε-neighbourhood of $\overline{\phi}(x)$ in X. Let $V(x, \delta) = \{z \in \overline{T} \mid d(z,x) < \delta\}$. Then $\widetilde{\phi}(V(x,\delta)) \subseteq V(\widetilde{\phi}(x), \varepsilon)$. Indeed, for any $z = (z_1, z_2, z_3) \in V(x, \delta) \cap T$, at least two of the distances $\{d(z_1, x), d(z_2, x), d(z_3, x)\}$ are less than δ. Hence, continuity of $\overline{\phi}$ implies that at least two of three distances $\{d(\overline{\phi}(z_i), \overline{\phi}(x))\}_{i=1,2,3}$ are less than ε. On the other hand, by the continuity of $\overline{\phi}$, all elements in $V(x,\delta) \cap X$ are also mapped by $\widetilde{\phi}$ into $V(\widetilde{\phi}(x), \varepsilon)$.

Now if $x \in \mathrm{Fix}\overline{\phi}$, then for every $g \in \mathrm{Fix}\phi$ we have $\overline{\phi}(gx) = \phi(g)\overline{\phi}(x) = gx$ and so $\mathrm{Fix}\overline{\phi}$ is $\mathrm{Fix}\phi$-invariant. Moreover, for any sequence (x_n) of elements of $\mathrm{Fix}\overline{\phi}$ with $x_n \to x \in X$ we have that $\overline{\phi}(x_n) \to \overline{\phi}(x)$ by continuity and hence $x \in \mathrm{Fix}\overline{\phi}$. Consequently, $\mathrm{Fix}\overline{\phi}$ is a closed $\mathrm{Fix}\phi$-invariant subset of X. Hence, $\Lambda \mathrm{Fix}\phi \subseteq \mathrm{Fix}\overline{\phi}$, by the definition of $\Lambda \mathrm{Fix}\phi$. □

3. The main result

In the present section we show that if G is a non-elementary convergence group, acting geometrically finite on a compact metric space X and ϕ a symmetric endomorphism of G, then $\Lambda \mathrm{Fix}\phi$ consists only of conical limit points and bounded parabolic points. In fact, we shall show that the quotient of $T' = (\Lambda \mathrm{Fix}\phi)^3 \setminus \Delta$, where Δ is the diagonal, under the action of $\Lambda \mathrm{Fix}\phi$ is the union of a compact set and a finite number of $\Lambda \mathrm{Fix}\phi$-quotients of cusp neighbourhoods of bounded parabolic points of $\Lambda \mathrm{Fix}\phi$. For notational simplicity, let us denote $\overline{T}' = T' \cup \Lambda \mathrm{Fix}\phi \subseteq \overline{T}$.

Assume that p is a bounded parabolic point of X. Then G_p acts properly discontinuously and cocompactly on $X \setminus \{p\}$ ([14, Theorem 3A]). Let C be a

compact set such that $G_pC = X \setminus \{p\}$ and W a compact neighbourhood of C in $\overline{T}$ not containing p. Define a *cusp neighbourhood* of p to be an open subset of T of the form $T \setminus G_pW$. A cusp neighbourhood U_p of p is G_p-invariant. A cusp neighbourhood is called *precisely invariant* if $gU_p = U_p$ for every $g \in G_p$ and $gU_p \cap U_p = \emptyset$ if $g \in G \setminus G_p$. Lemma 3F in [14] proves the existence of precisely invariant cusp neighbourhoods for every parabolic point of G. Let $\mathrm{cl}_T U_p$ denote the closure in T of U_p. Then, by making U_p small enough we may assume that $\mathrm{cl}_T U_p$ is also precisely invariant (again, see [14]).

In [14, Theorem 1B] it is shown that G is relatively hyperbolic if and only if T/G is the union of a compact set and a finite number of G-quotients of cusp neighbourhoods of bounded parabolic points. We shall make extensive use of this result in what follows.

Lemma 2. *With the above hypotheses, $T' = (\mathrm{Fix}\phi)P' \cup (\bigcup (U_{q_i} \cap T'))$ for some compact subset $P' \subseteq T'$ and a collection $\{q_i\}$ of bounded parabolic points of G such that $\{q_i\} \subset \Lambda\mathrm{Fix}\phi$.*

Proof. From hypotheses we know that $T = GP \cup (\bigcup U_{p_i})$ where P is a compact subset of T and U_{p_i} are cusp neighbourhoods of bounded parabolic points p_i in T. Without loss of generality we may assume that $U_{p_i} \cap T' = \emptyset$ whenever p_i is not a limit point of $\Lambda\mathrm{Fix}\phi$. The lemma will be proved by constructing a compact subset P' of T' and a cusp neighborhood U'_{p_i} for each parabolic point p_i which is also a limit point of $\Lambda\mathrm{Fix}\phi$ such that $GP \cap T' = (\Lambda\mathrm{Fix}\phi)P'$ and $U'_{p_i} = U_{p_i} \cap T'$.

Let $G = \bigcup_{i\in I} (\mathrm{Fix}\phi) g_i$ for some right coset representative system $\{g_i\}_{i\in I}$ of $\mathrm{Fix}\phi$ in G. Since $\widehat{\phi}$ is a continuous function, we have that $\widehat{\phi}(P)$ is a compact subset of T. Define $P_\phi = P \cup \widehat{\phi}(P)$. We claim that there are finitely many g_i such that $g_iP \cap T' \neq \emptyset$. To see that, let $x \in P$ such that $g_ix \in T'$. So $g_ix \in \mathrm{Fix}\widehat{\phi}$ and so $\widehat{\phi}(g_ix) = g_ix$. Hence, $\phi(g_i)\widehat{\phi}(x) = g_ix$ or equivalently $g_i^{-1}\phi(g_i)\widehat{\phi}(x) = x$. But $\widehat{\phi}(x) \in \widehat{\phi}(P)$ and so $g_i^{-1}\phi(g_i) \in S$ where

$$S = \{g \in G \mid gP_\phi \cap P_\phi \neq \emptyset\}.$$

But G acts properly discontinuously on T and so S is a finite set. Moreover, if $g_i^{-1}\phi(g_i) = g_j^{-1}\phi(g_j)$ then $g_jg_i^{-1} \in \mathrm{Fix}\phi$ and so $(\mathrm{Fix}\phi)g_i = (\mathrm{Fix}\phi)g_j$, thus $g_i = g_j$. This proves the claim.

Let $g_1, \ldots, g_k \in \{g_i\}_{i\in I}$ such that $g_iP \cap T' \neq \emptyset$ and

$$P' = (g_1P \cup \ldots \cup g_kP) \cap T'.$$

By its choice, P' is a compact subset of T'. This is the case since $\overline{T}'$ is closed in $\overline{T}$ and $(g_1P \cup \ldots \cup g_kP) \cap T' = (g_1P \cup \ldots \cup g_kP) \cap \overline{T}'$. We show that

$$(\mathrm{Fix}\phi)P' = GP \cap T'.$$

It is obvious that $(\mathrm{Fix}\phi)P' \subseteq GP \cap T'$. Let $y \in GP \cap T'$. Then $y = gx$ for some $x \in P$. On the other hand, $g = hg_\lambda$ for some $h \in \mathrm{Fix}\phi$, and so $h^{-1}y = g_\lambda x$. Therefore, $g_\lambda x \in g_\lambda P$ and $h^{-1}y \in T'$ since T' is invariant under the action of $\mathrm{Fix}\phi$. Hence, $g_\lambda x \in g_\lambda P \cap T'$ and so $g_\lambda \in \{g_1, \ldots, g_k\}$. Thus, $y = hv_ix$ which implies that $y \in (\mathrm{Fix}\phi)P'$.

Next, using similar arguments, we show that the same properties are true for the cusp neighbourhoods of every bounded parabolic point of G that is also a point of $\Lambda\text{Fix}\phi$. Indeed, let p be a bounded parabolic point of ΛG such that $p \in \Lambda\text{Fix}\phi$. Choose C a compact subset of $\Lambda G \setminus \{p\}$ such that $G_pC = \Lambda G \setminus \{p\}$. Then take a compact neighbourhood W of C in $\overline{T} \setminus \{p\}$. The open subset $U_p = T \setminus G_pW$ is a cusp neighbourhood of p.

Let $H = G_p \cap \text{Fix}\phi$ and assume that $G_p = \bigcup_{i\in I} Hq_i$ for some right coset representative system $\{q_i\}_{i\in I}$. Since C is compact and $\overline{\phi}$ a continuous function of X, again, $\overline{\phi}(C)$ is a compact subset of X. Define $C_\phi = C \cup \overline{\phi}(C)$. To show that there are finitely many q_i such that $q_iC \cap \Lambda\text{Fix}\phi \setminus \{p\} \neq \emptyset$, let $x \in C$ such that $q_ix \in \Lambda\text{Fix}\phi \setminus \{p\}$. Then $q_ix \in \text{Fix}\overline{\phi}$ and so $\overline{\phi}(q_ix) = q_ix$, i.e. $\phi(q_i)\overline{\phi}(x) = q_ix$ or $q_i^{-1}\phi(q_i)\overline{\phi}(x) = x$. But $\overline{\phi}(x) \in \overline{\phi}(C)$ and so

$$q_i^{-1}\phi(q_i) \in S_1 = \{g \in G_p \mid gC_\phi \cap C_\phi \neq \emptyset\}.$$

But G_p acts properly discontinuously on $\Lambda G \setminus \{p\}$ ([14, Theorem 3A]) hence S_1 is a finite set. Moreover, if $q_i^{-1}\phi(q_i) = q_j^{-1}\phi(q_j)$ then $q_jq_i^{-1} \in \text{Fix}\phi \cap G_p$ and so $Hq_i = Hq_j$ so $q_i = q_j$. Let $q_1, \dots, q_r$ be the elements of $\{q_i\}_{i\in I}$ for which $q_iC \cap \Lambda G \setminus \{p\} \neq \emptyset$ for $i = 1, \dots, r$ and denote C' the set

$$C' = (q_1C \cup \ldots \cup q_rC) \cap \Lambda\text{Fix}\phi \setminus \{p\}.$$

It is obvious that C' is a compact subset of $\Lambda\text{Fix}\phi$, not containing p since $\Lambda\text{Fix}\phi$ is a closed subset of ΛG and

$$(q_1C \cup \ldots \cup q_rC) \cap \Lambda\text{Fix}\phi \setminus \{p\} = (q_1C \cup \ldots \cup q_rC) \cap \Lambda\text{Fix}\phi.$$

Again,

$$HC' = G_pC \cap \Lambda\text{Fix}\phi \setminus \{p\} = \Lambda\text{Fix}\phi \setminus \{p\}.$$

Indeed, let $y \in G_pC \cap \Lambda\text{Fix}\phi \setminus \{p\}$. Then $y = gx$ for some $x \in C$ and some $g \in G_p$. Again $g = hq_\lambda$ for $h \in (G_p \cap \text{Fix}\phi)$ and $h^{-1}y = q_\lambda x$. So $q_\lambda x \in q_\lambda C \cap \Lambda\text{Fix}\phi \setminus \{p\}$. Hence $q_\lambda \in \{q_1, \dots, q_r\}$ and $y = hq_ix \in (G_p \cap \text{Fix}\phi)C' = HC'$. Hence, C' is a compact subset of $\Lambda\text{Fix}\phi \setminus \{p\}$ such that $(G_p \cap \text{Fix}\phi)C' = HC' = \Lambda\text{Fix}\phi \setminus \{p\}$.

Now we use the compact set C' to construct the cusp neighborhood U'_p as follows. Let $M = q_1W \cup \ldots \cup q_rW$. Then M is a compact subset of $\overline{T}$ that belongs entirely into $\overline{T} \setminus \{p\}$. We claim again that there are finitely many $t_i \in \{q_i\}_{i\in I}$ such that $t_iM \cap \overline{T}' \setminus \{p\} \neq \emptyset$. Define $M_\phi = M \cup \widetilde{\phi}(M)$. M_ϕ is compact, since $\widetilde{\phi}$ is a continuous function. If $x \in M$ with $t_ix \in \overline{T}' \setminus \{p\}$, then $t_ix \in \text{Fix}\widetilde{\phi}$ and $\widetilde{\phi}(t_ix) = t_ix$, i.e. $t_i^{-1}\phi(t_i)\widetilde{\phi}(x) = x$. But $\widetilde{\phi}(x) \in \widetilde{\phi}(M)$ and so $t_i^{-1}\phi(t_i) \in \{g \in G_p \mid gM_\phi \cap M_\phi \neq \emptyset\}$ which is a finite set by the discontinuity of the action of G_p on $\overline{T} \setminus \{p\}$ ([14, Theorem 3A]). Moreover, $t_i^{-1}\phi(t_i) = t_j^{-1}\phi(t_j)$ implies that $t_jt_i^{-1} \in G_p \cap \text{Fix}\phi = H$ and so $Ht_i = Ht_j$, i.e. $t_i = t_j$. Let $t_1, \dots, t_n$ the elements of $\{q_i\}_{i\in I}$ for which $t_iM \cap \overline{T}' \setminus \{p\} \neq \emptyset$.

Let

$$W' = (t_1M \cup \ldots \cup t_nM) \cap \overline{T}' \setminus \{p\}.$$

It is apparent that

$$W' = (r_1W \cup \ldots \cup r_mW) \cap \overline{T}' \setminus \{p\}$$

with $r_i \in \{q_i\}_{i\in I}$ and r_iW are the only "copies" of W for which we might have $r_iW \cap \overline{T}' \setminus \{p\} \neq \emptyset$. Define $W'' = (v_1W \cup \ldots \cup v_\mu W) \cap \overline{T}' \setminus \{p\}$ where $v_i \in \{r_1, \ldots, r_m\}$ and $v_iW \cap \overline{T}' \setminus \{p\} \neq \emptyset$.

We have

$$HW'' = G_pW \cap \overline{T}' \setminus \{p\}.$$

For $y \in G_pW \cap \overline{T}' \setminus \{p\}$ we have $g \in G_p$ and $x \in W$ such that $y = gx$. Again, $g = hq_\lambda$ with $h \in H$ and $h^{-1}y = q_\lambda x$. So $q_\lambda x \in q_\lambda W \cap \overline{T}' \setminus \{p\}$ and so $q_\lambda \in \{v_1, \ldots, v_\mu\}$ and $y = hr_ix \in HW''$.

Now $\{v_1, \ldots, v_\mu\} = \{q_1, \ldots, q_r\} \cup \{q_{r+1}, \ldots, q_a\}$. Since $q_iC \cap \Lambda\mathrm{Fix}\phi \setminus \{p\} = \emptyset$ for all $i \in \{q_{r+1}, \ldots, q_a\}$ but $q_iW \cap \overline{T}' \setminus \{p\} \neq \emptyset$, W is a compact subset and $\overline{T}'$ is closed in $\overline{T}$, we have that $q_iW \cap \overline{T}' \setminus \{p\}$ is a compact subset of T'. Hence, we may consider W'' as a compact neighbourhood of C' in $\overline{T}'$ not containing p.

Take

$$U'_p = T' \setminus (G_p \cap \mathrm{Fix}\phi)W''.$$

Then, by its construction, U'_p is a cusp neighbourhood of $p \in \overline{T}'$ such that $U'_p = U_p \cap T'$. □

The following is an immediate consequence of the above proof.

Corollary 1. *Every bounded parabolic point of G that is also an element in $\Lambda\mathrm{Fix}\phi$ is a bounded parabolic point of* $\mathrm{Fix}\phi$.

Lemma 3. *With the above hypotheses, $T'/\mathrm{Fix}\phi$ is the union of a compact subset P' and a finite number of* $\mathrm{Fix}\phi$*-quotients of cusp neighbourhoods of bounded parabolic points of* $\Lambda\mathrm{Fix}\phi$.

Proof. Assume that $\{U_{p_i}\}$ are chosen such that $\mathrm{cl}_T U_{p_i}$ are precisely invariant. Assume further that $T'/\mathrm{Fix}\phi$ contains an infinite number of $\mathrm{Fix}\phi$-quotients of cusp neighbourhoods $\{U'_p\}$ as constructed in Lemma 2. Since every U'_p is induced by some U_p, there is a bounded parabolic point $p \in \Lambda\mathrm{Fix}\phi$ and infinitely many $g_i \in \mathrm{Fix}\phi$ such that $g_iU_p/\mathrm{Fix}\phi$ are different in $T'/\mathrm{Fix}\phi$. This means that g_i is a sequence of discrete elements of $\mathrm{Fix}\phi$ and so there is a subsequence, denoted again g_i, such that $g_ip \to b$ for some $b \in X$. But by Lemma 3E [14], g_iU_p accumulate, a contradiction to the precisely invariant choice of $\{U_{p_i}\}$. □

Now we are able to prove our main theorem.

Theorem 1. *Let G be geometrically finite group and ϕ a symmetric endomorphism of G. Then* $\mathrm{Fix}\phi$ *is geometrically finite.*

Proof. If $\Lambda\mathrm{Fix}\phi = \emptyset$, then $\mathrm{Fix}\phi$ is finite and hence quasiconvex in a trivial way. If $\Lambda\mathrm{Fix}\phi$ has one point then $\mathrm{Fix}\phi$ is parabolic and is relatively quasiconvex. If $\mathrm{Fix}\phi$ has two points then $\mathrm{Fix}\phi$ is cyclic-by-finite, and is contained as a subgroup of finite index into a unique maximal cyclic-by-finite elementary subgroup (see [13]). Hence, $\mathrm{Fix}\phi$ is quasiconvex in G by the result of Osin [11, Corollary 1.7].

So we may assume that $\Lambda\mathrm{Fix}\phi$ has at least three points. Then the result is an immediate consequence of Lemmas 2 and 3. □

4. Applications

Let G be a finitely generated relatively hyperbolic group and ϕ a monomorphism of G which maps maximal parabolic subgroups into conjugates of themselves, i.e. for each maximal parabolic subgroup G_p there is a maximal parabolic subgroup G_q such that $\phi(G_p) \subset G_q$. In the special case where ϕ is an automorphism permuting maximal parabolic subgroups, the result of Bowditch [2, Theorem 9.4] shows that ϕ is symmetric. In general, in order to show that ϕ is symmetric we assume further that its image is a relatively quasiconvex subgroup of G. What shall need the following result.

Theorem 2 ([8, Theorem 1.1])**.** *Let G be a countable group and X, Y compact metrizable spaces endowed with geometrically finite actions of G. If each maximal parabolic subgroup with respect to X is a parabolic subgroup with respect to Y, then there exists a continuous map $\pi : X \to Y$ such that $\pi(gx) = g\pi(x)$ for each $g \in G$ and $x \in X$.*

Theorem 3. *Let G be a finitely generated group acting minimally and geometrically finite on a compact metrizable space X. Suppose that ϕ is a monomorphism of G such that $\phi(G)$ is a geometrically finite subgroup of G. If ϕ maps maximal parabolic subgroups into conjugates of themselves, then ϕ is symmetric. In particular,* Fixϕ *is geometrically finite.*

Proof. We consider the action of G on $\Lambda\phi(G)$ giving by $g * x = \phi(g)x$. Since ϕ is monomorphism and $\phi(G)$ is geometrically finite, it follows that the twisted action of G on $\Lambda\phi(G)$ is geometrically finite. Moreover, the maximal parabolic subgroup corresponding to a parabolic point $p \in \Lambda\phi(G)$ is $\phi^{-1}(G_p)$, where G_p is the parabolic subgroup of the initial action of G on $X = \Lambda G$. Thus the hypotheses of the above theorem are satisfied and we have a continuous map $\pi : \Lambda G \to \Lambda\phi(G) \subseteq \Lambda G$ such that $\pi(gx) = \phi(g)\pi(x)$. This proves that ϕ is symmetric and therefore Fixϕ is geometrically finite. ☐

An immediate consequence of Theorem 3 is the following result by Minasyan and Osin.

Corollary 2 ([7])**.** *Let G be a finitely generated relatively hyperbolic group and ϕ a symmetric automorphism of G. Then* Fixϕ *is relatively quasiconvex.*

A group G is *locally quasiconvex* if all its finitely generated subgroups are quasiconvex with respect to some finite generating set of G. Locally quasiconvex subgroup are studied in [4]. The family of locally quasiconvex groups contains free groups, surface groups, certain Coxeter groups and certain small cancellation groups. Moreover, as shown in [4] it is closed under free products and certain amalgamated free products. Theorem 3 has the following consequence concerning locally quasiconvex hyperbolic groups.

Corollary 3. *If G is a locally quasiconvex hyperbolic group, then* Fixϕ *is quasiconvex for any monomorphism ϕ of G.*

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